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ESTIMATES OF THE TRAPPED-MODE FREQUENCIES OF OSCILLATION OF A LIQUID IN THE PRESENCE OF SUBMERGED BODIES†

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The problem of the steady oscillations of an unbounded liquid with a free surface in the presence of a system of submerged horizontal cylindrical bodies of arbitrary cross-section is considered. The problem describes the interaction of bodies with waves which propagate at an arbitrary angle to the generatrix of the system of bodies and, also, the oscillations of a liquid in a channel with vertical walls. The criterion of uniqueness, proposed earlier in [1], is developed to find estimates of the trapped-mode frequencies. Estimates are obtained of the frequencies which are both in the continuous spectrum of the problem as well as outside it. Results of calculations are presented and a comparison is made with existing estimates.

1. FORMULATION OF THE PROBLEM

The problem of the steady oscillations of an ideal incompressible heavy liquid of infinite depth with a free surface in the presence of fully submerged cylindrical bodies of arbitrary cross-section, is considered. It is assumed that the motion of the liquid is irrotational and harmonic in time with a frequency ω. The linear approximation of the theory of surface waves is used. Here, the boundary conditions are referred to the unperturbed free surface.

We shall consider two situations: (1) the liquid is unbounded in the horizontal directions and the generatrix of the system of cylindrical bodies z makes and arbitrary non-zero angle θ with the direction of propagation of the surface waves, (2) the liquid is confined in a channel with vertical walls which are orthogonal to the generatrix of the system of cylindrical bodies. The notation is introduced in Fig. 1, where a section in a plane which is orthogonal to the generatrix of the cylinders is shown: W is the domain occupied by the liquid, S is the wetted surface of the bodies, F is the free surface $y = 0$ and $\mathbf{n}$ is the unit normal.

We will first consider the case of a liquid which is unbounded in the horizontal directions. It is assumed that the motion of the liquid is periodic in the variable z, and it is then described by a velocity potential $\mathrm{Re}\{u(x,y)e^{\pm ikz}e^{-i\omega t}\}$, where $k = \nu\sin\theta$ is the projection of the wave vector onto the z axis, $\nu = \omega^2/g$ is the wave number and g is the acceleration due to gravity. The potential u satisfies the following boundary-value problem

$$(\nabla^2 - k^2)u = 0 \text{ in } W \tag{1.1}$$

$$\partial_y u - \nu u = 0 \text{ in } F \tag{1.2}$$

$$\partial_n u = f \text{ in } S \tag{1.3}$$

$$|\nabla u| = O(|x+iy|^M) \text{ when } |x+iy| \to \infty \tag{1.4}$$

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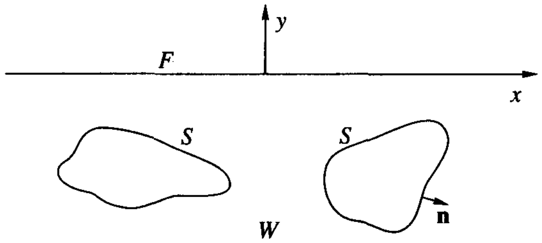

Fig. 1

(here and henceforth $\partial_a = \partial/\partial a$). Condition (1.4) must be uniformly satisfied with respect to $\arg(x + iy)$ for a certain $M > 0$. The function f in condition (1.3) is determined by the type of problem (radiation or diffraction). In order to describe the process of the radiation or scattering of waves, problem (1.1)–(1.4) must be augmented with the conditions which determine the form of the wave motion at infinity

$$\int\limits_{W \cap \{|x| = r\}} |\partial_{|x|} u - ilu|^2 dy = o(1) \text{ when } r \to \infty; \quad l = \sqrt{\nu^2 - k^2} \tag{1.5}$$

The case when $\nu = k$ (the passage of a wave along the generatrix of the cylinders) must be excluded from the treatment (see [2]).

This problem also describes the oscillations of a liquid in a channel of infinite depth with vertical walls $z = \pm b$, when the generatrix of the system of cylindrical bodies is orthogonal to the walls. In this case, the motion of the liquid is described by the velocity potential $\mathrm{Re}\{u(x, y)e^{-i\omega t}\} \cos kz$, where $kb = n\pi$ ($n = 1, 2, \ldots$) or the potential $\mathrm{Re}\{u(x, y)^{-i\omega t}\} \sin kz$, where $kb = (2n - 1)\pi/2$ ($n = 1, 2, \ldots$) (in this case, the condition $\partial_n u = 0$ is satisfied on the walls). Unlike a liquid which is boundless in the horizontal directions, a combination of parameters $\nu < k$ is possible in this case.

An expansion at infinity has been obtain [3, pp. 815, 822] in the form of a linear combination of regular wave solutions $e^{\nu y}\sin lx$, $e^{\nu y}\cos lx$, a source (see (2.1)), a horizontal dipole and a series consisting of wave-free solutions (see (2.7)), where $x_0 = 0, y_0 = 0$) for any potential which satisfies Eq. (1.1) and condition (1.2) outside a certain semicircle with its centre in the free surface. In the case when $\nu < k$, the source and dipole potentials decrease at infinity and there is no wave term. Hence, when $\nu < k$, the solution of problem (1.1)–(1.4) has an estimate at infinity $O(|x + iy|^{-n})$ for any $n > 0$.

It is well known (for example, see [4, § 2.2.1]) that the energy functional

$$\int\limits_W (|\nabla u|^2 + k^2|u|^2)dxdy < \infty \tag{1.6}$$

is bounded in the case of the solutions of the homogeneous problem ($f = 0$) when $\nu > k$. Hence, the uniqueness of the solution of the homogeneous problem (1.1)–(1.5) when $\nu > k$ and of problem (1.1)–91.4) when $\nu < k$ is equivalent to the absence of solutions of problem (1.1)–(1.3), (1.6) when $f = 0$. Without loss in generality, it can be assumed that the solution of the homogeneous problem is a real potential.

The homogeneous problem (1.1)–(1.3), (1.6) ($f = 0$) can be considered as a problem involving finding a point spectrum, where ν is an eigenvalue and u is an eigenfunction or, in other words, a trapped mode of oscillation. Since, when $\nu > k$, problem (1.1), (1.2) admits of a solution with *infinite* energy (due to wave formation at infinity) in this case it is possible to speak of a point spectrum which is imbedded in a continuous spectrum.

The aim of this paper is to find the subsets of the parameter space in which problem (1.1)–(1.3), (1.6) when $f = 0$ does not have trivial solutions or, to put it differently, to find estimates of the trapped-mode frequencies. The approach proposed in [1] for $k = 0$ is developed for this purpose. It will be shown that the method is applicable in the case when $k \neq 0$ and that it enables one to obtain estimates of the trapped-mode frequencies both in the continuous spectrum as well as outside of it.

A fairly complete review of papers dealing with the question of the existence of trapped modes and estimates of the corresponding frequencies is available [4]. In the case of the problem being considered, the majority of the results refer to the special case when $k = 0$ when the motion of the liquid is described by Laplace's equation.

The existence of trapped modes when $\nu > k$ in the general case when $k \neq 0$ has only been established for partially submerged bodies (see [5]), although it is possible that the approach, which was used to obtain examples of non-uniqueness [6] in the case of completely submerged bodies when $k = 0$, can be generalized to construct examples in the case when $k \neq 0$.

The existence of solutions of the homogeneous problem (1.1)–(1.3), (1.6) has been established [7–9] for submerged bodies of arbitrary form in the range $\nu < k$, and theorems for comparing the eigenvalues for the embedded domains (the principle of monotonicity) have also been proved [9]. A numerical investigation of the symmetric trapped modes for circular cylinders has been carried in [10] and formulae have been derived for estimates of the lower limit of the trapped-mode frequencies. More universal approaches to obtaining estimates have been proposed in [11, 12].

The problem of uniqueness when $\nu > k$ as been far better studied for the special case when $k = 0$ and for partially submerged bodies when $k \neq 0$ (see [4, Ch. 1, 13]). There are two basic schemes for proving uniqueness: the John scheme (1950, see [4, section 3.2] and, also, [11]) and the so-called Maz'ya identity (1973) (for a detailed description, see [4, section 2.2]). Both schemes impose substantial geometrical constraints. The John scheme is ineffective in the case of bodies which are fairly remote from one another in a horizontal direction. The scheme based on the Maz'ya identity is in a certain sense the opposite. Having found a vector field with the required characteristics, it is possible to produce certain configurations of the bodies which do not sustain trapped modes.

A new scheme [1] for proving uniqueness and for obtaining estimates of the trapped-mode frequencies has recently been proposed. The method is based on the combined use of Green's identity and the maximum principle for elliptic operators and it enables one to consider systems of any number of bodies of arbitrary geometry. At the same time, there is no characteristic constraint for schemes based on the John approach associated with the distance between the bodies.

The criterion of uniqueness [1] takes the form of an inequality which includes an integral over the wetted surface, and the integrand includes arbitrary Green's functions. The inequality can be checked numerically. At the same time, estimates of the gradient of Green's function enable one to obtain simple estimates of the uniqueness domain in the parameter space of the problem, the frequency (or wave number) and the depth of immersion of the system of bodies. In particular, the estimates guarantee that there are no trapped modes for a sufficiently great depth of immersion or sufficiently high frequency values. Below, it will be shown that the method provides estimates of the trapped-mode frequencies both in the continuous spectrum when $\nu > k$ (where the method has no analogous when $k \neq 0$) as well as in the range where $\nu < k$ (where the method has certain advantages and disadvantages compared with the corresponding estimates in [11, 12]).

2. GREEN'S FUNCTION AND THE CRITERION OF UNIQUENESS

We will use the notation $G(x, y; x_0, y_0)$ for the potential of a source located at the point (x_0, y_0) and calculated at a point (x, y). Then, when $y_0 < 0$, the potential G, as a function of x and y, satisfies the equations

$$\nabla^2_{(x,y)}G - k^2G = -\delta(x - x_0, y - y_0), \quad y_0 < 0$$
$$\partial_y G - \nu G = 0, \quad y = 0$$

where δ is the delta-function. Moreover, the potential G must be bounded in the domain $y < 0$ with the neighbourhood of the point (x_0, y_0) cut out.

We have (see [7], for example)

$$G(x, y; x_0, y_0) = \frac{K_0(kr_-) - K_0(kr_+)}{2\pi} + \frac{1}{\pi}\int_0^\infty H(y + y_0, t)\cos(x - x_0)t\,dt \tag{2.1}$$

where

$$r_\pm = [(x - x_0)^2 + (y \pm y_0)^2]^{1/2}$$
$$K_n(z) = \int_0^\infty e^{-z\operatorname{ch}\mu}\operatorname{ch}n\mu\,d\mu, \quad H(z, t) = \frac{e^{z\sqrt{t^2+k^2}}}{\sqrt{t^2+k^2} - \nu} \tag{2.2}$$

($K_n(z)$ is the modified Bessel function, see [14], for example, formula 8.432.1). The integrand in formula (2.1) has a first order pole when $\nu > k$ and, in this case, the integral is evaluated in the sense of the principal value.

We now apply the third Green's identity to the hypothetical trapped mode, that is, to the solution of the homogeneous problem (1.1)–(1.3), (1.6) and to Green's function G. Taking account of the condition on the free surface and the boundedness of the potential and its derivatives at infinity, we find

$$u(x_0, y_0) = \int_S u(x, y)\partial_{n(x, y)}G(x, y; x_0, y_0)ds_{(x, y)}, \quad (x_0, y_0) \in W \tag{2.3}$$

It follows from the last formula that

$$|u(x_0, y_0)| \le \max_{(x, y) \in S}\{|u(x, y)|\}\int_S |\partial_{n(x, y)}G(x, y; x_0, y_0)|ds_{(x, y)}$$

Note that Green's function can be analytically extended form the domain $y < 0$ into the domain $y < -y_0$ (this follows, in particular, from equality (2.1)). Hence, representation (2.3) is also suitable for points (x_0, y_0) belonging to the free surface. We therefore conclude that

$$\sup_F|u| \le \max_S|u| \sup_{(x_0, y_0) \in F}\left\{\int_S |\partial_{n(x, y)}G(x, y; x_0, y_0)|ds_{(x, y)}\right\} \tag{2.4}$$

We now compare the quantities $\sup_F|u|$ and $\max_S|u|$. We first note that Hopf's powerful maximum principle (see [15, theorem (3.5)], for example) guarantees that the potential u, which is not constant and satisfies Eq. (1.1), cannot reach a non-positive minimum and non-negative maximum at an internal point of any finite subset of W. This assertion also holds for the whole of the unbounded domain W and it is sufficient to note that the potential of a trapped mode decreases at infinity.

Suppose the wetted surface of the bodies S is fairly regular, that is, it belongs to the class C^1, and that the "condition of an internal sphere" (the point must lie on the boundary of a circle located in the domain W) is satisfied at all of its points. In particular, contours of the class C^2 satisfy these conditions. With these constraints, the assertion [15, Lemma 3.4] holds for any finite domain $W_0 \subset W$ which guarantees that, at the point on the boundary at which u reaches a maximum non-negative (minimum non-positive) value, any derivative along a direction which is external with respect to the domain is strictly positive (negative).

We now consider the domain $W_R = W \cap \{|x^2 + y^2| \le R\}$. It is obviously possible to find a sufficiently large value of R such that the maximum value of $|u|$ is reached on the surface of the bodies S or on a section of the free surface $F_R = F \cap \{|x| \le R\}$. We also note that, if the maximum in $|u|$ corresponds to the maximum of the function u, then this maximum is positive and, if it corresponds to the minimum, then this minimum value is negative. Consequently, it is possible to make use of the assertion which has been presented above, according to which $\max|u|$ in the domain W_R cannot be reached on the surface S since this contradicts condition (1.3). It is therefore established that either $u = 0$ in the domain W or

$$\sup_F|u| > \max_S|u|$$

On combining the last inequality with (2.4), we arrive at the following assertion: if the inequality

$$\sup_{(x_0, y_0) \in F}\left\{\int_S |\partial_{n(x, y)}G(\nu, k; x, y; x_0, y_0)|ds_{(x, y)}\right\} < 1 \tag{2.5}$$

is satisfied for a given configuration of the bodies S and specified values of the parameter ν and k, then the homogeneous problem (1.1)–(1.3), (1.6) has only a trivial solution. It has been shown in [1] that this assertion can be extended to the case of contours with corner points and cuspidal points protruding into the liquid. It can also be proved using the well-known results in [16] that the last assertion holds for configurations which satisfy an external cone condition. In particular, corner points protruding into the interior of bodies are permissible.

Following the approach described earlier in [1], it can be shown that the criterion presented can be strengthened using auxiliary potentials Φ_i which satisfy relations (1.1) and (1.2), and criterion (2.5) can be replaced by the inequality

$$\sup_{(x_0,\,y_0)\in F}\left\{\int_S\left|\partial_{n(x,\,y)}G(x,y;x_0,y_0)+\sum_i a_i(x_0,y_0)\partial_n\Phi_i(x,y;x_0,y_0)\right|ds_{(x,\,y)}\right\}\le 1 \tag{2.6}$$

where $a_i = a_i(x_0, y_0)$ are coefficients which can be determined using the minimization problem

$$\min_{a_1,\,a_2,\,\ldots}\int_S\left|\partial_{n(x,\,y)}G(x,y;x_0,y_0)+\sum_i a_i\partial_n\Phi_i(x,y;x_0,y_0)\right|ds_{(x,\,y)},\quad (x_0,y_0)\in F$$

This is a problem of linear programming, and there are efficient algorithms for its numerical solution (the simplex method). The simple wave solutions $e^{\nu y}e^{\pm ilx}$ and the singular solutions of problem (1.1), (1.2) with the discontinuity located within the bodies, that is, multipoles or simpler wave-free singular potentials $\operatorname{Re}\{\Psi_n(x, y; x_0, y_0)\}$, $\operatorname{Im}\{\Psi_n(x, y; x_0, y_0)\}$, can be used as the auxiliary potentials $\Phi_i(x, y; x_0, y_0)$. Here,

$$\begin{aligned}\Psi_n(x,y;x_0,y_0) &= \nu K_n(kr_-)e^{in\tau_-}-\nu K_n(kr_+)e^{in\tau_+}-\\ &-\frac{k}{2}\left[\sum_\pm K_{n\pm1}(kr_-)e^{i(n\pm1)\tau_-}-\sum_\pm K_{n\pm1}(kr_+)e^{i(n\pm1)\tau_+}\right],\quad n=0,1,2,\ldots\end{aligned} \tag{2.7}$$

$r_\pm$ are defined by formulae (2.2), and $\tau_\pm$ are defined by the relations

$$x-x_0 = r_-\sin\tau_- = r_+\sin\tau_+,\quad y = r_-\cos\tau_-+y_0 = -r_+\cos\tau_+-y_0$$

Note that the quantity sup in formulae (2.5) and (2.6) is defined in the unbounded set F which, of course, is unacceptable from the point of view of the applicability of the criterion for numerical implementation. The part of the free surface, in which it suffices to verify that criterion (2.6) is satisfied, can be found using estimates of the gradient of Green's function which will be obtained below. This has been described earlier in greater detail in [1] where examples of the numerical investigation of the problem of uniqueness on the basis of criterion (2.6) for $k = 0$ can be found.

3. ESTIMATES OF THE DERIVATIVES OF GREEN'S FUNCTION

Using representation (2.1) and introducing the notation $G_0 = G(x, y; x_0, 0)$, we find

$$\begin{aligned}\partial_x G_0 &= -\frac{1}{\pi}\int_0^\infty tH(y,t)\sin(x-x_0)t\,dt\\ \partial_y G_0 &= \frac{1}{\pi}\int_0^\infty\sqrt{t^2+k^2}H(y,t)\cos(x-x_0)t\,dt\end{aligned} \tag{3.1}$$

(the integrals are evaluated in the sense of the principal value when $\nu > k$).

We make the replacement of variables [3, Section 2] and write

$$\partial_y G_0 = \frac{k}{2\pi}\int_{-\infty}^{\infty}J(\mu)\operatorname{ch}^2\mu\,d\mu,\quad J(\mu) = \frac{\exp\{-kr_-\operatorname{ch}(\mu-i\varphi)\}}{\operatorname{ch}\mu-\lambda},\quad \lambda = \frac{\nu}{k} \tag{3.2}$$

The quantity φ is defined by the relation $\operatorname{tg}\varphi = (x-x_0)/y$ such that $\varphi\in(-\pi/2, \pi/2)$ when $y < 0$. In the case when $\nu > k$, the integrand has two poles on the real axis and, when $\nu < k$, the poles are located on the imaginary axis at the points $\mu = \pm i(\arccos(\lambda) + 2\pi n)$ $(n = 0, 1, 2, \ldots)$. Later in this section, we will consider values of $\lambda > 1/\sqrt{2}$.

We will first return to the case when $\varphi \in [0, \pi/2)$. On shifting the contour of integration upwards ($\mu \to \mu + i\pi/4$), we find

$$\partial_y G_0 = \frac{k}{2\pi} I_y(x - x_0, y) + \nu^2 e^{\nu y} \begin{cases} l^{-1} \sin l(x - x_0) & \text{when } \lambda > 1 \\ l_*^{-1} e^{l_*(x - x_0)} & \text{when } \lambda \in (1/\sqrt{2}, 1) \end{cases} \tag{3.3}$$

where

$$l_* = \sqrt{k^2 - \nu^2}, \quad I_y(x - x_0, y) = \int_{-\infty}^{\infty} J(\mu + i\pi/4) \operatorname{ch}^2(\mu + i\pi/4) d\mu \tag{3.4}$$

We introduce the notation

$$\Phi_c(\mu) = kr_- \cos(\pi/4 - \varphi) \operatorname{ch}\mu, \quad \Phi_s(\mu) = kr_- \sin(\pi/4 - \varphi) \operatorname{sh}\mu$$

Using the last formula of (3.4), we find

$$I_y(x - x_0, y) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{f(\mu)}{g(\mu)} \exp\{-\Phi_c(\mu)\} d\mu$$

$$f(\mu) = (\sqrt{2} \operatorname{ch}\mu \operatorname{ch}2\mu - 2\lambda) \cos(\Phi_s(\mu)) + (\sqrt{2} \operatorname{ch}2\mu + 4\lambda \operatorname{ch}\mu) \operatorname{sh}\mu \sin(\Phi_s(\mu)) \tag{3.5}$$

$$g(\mu) = (\sqrt{2}\lambda - \operatorname{ch}\mu)^2 + \operatorname{sh}^2\mu > 0$$

It is obvious that

$$|f(\mu)| \le \sqrt{2} \operatorname{ch}2\mu \sqrt{\operatorname{ch}^2\mu + \operatorname{sh}^2\mu} + 2\lambda \sqrt{1 + 4\operatorname{ch}^2\mu \operatorname{sh}^2\mu} = \operatorname{ch}2\mu(\sqrt{2\operatorname{ch}2\mu} + 2\lambda)$$

Taking the relation $\operatorname{ch}2\mu = 2\operatorname{ch}^2\mu - 1$ into account, we have

$$\begin{gathered} |f(\mu)| \le \operatorname{ch}\mu f_1(\operatorname{ch}\mu), \quad |f(\mu)| \le \operatorname{ch}2\mu f_2(\operatorname{ch}\mu) \\ f_1(t) = 4t^2 + 4\lambda t - 2, \quad f_2(t) = 2t + 2\lambda \end{gathered} \tag{3.6}$$

Since $\operatorname{ch}\mu \ge 1$, it is convenient to make the change of variable $t \to z^2 + 1$ in f_1 and f_2 and $\operatorname{ch}\mu \to z^2 + 1$ in g. We obtain

$$f_1(z) = 4z^2 + 4(2 + \lambda)z^2 + 2 + 4\lambda, \quad f_2(z) = 2z^2 + 2\lambda + 2$$

$$g(z) = 2z^4 + 2(2 - \sqrt{2}\lambda)z^2 + 1 - 2\sqrt{2}\lambda + 2\lambda^2$$

We will seek the coefficients c_1 and c_2 such that $f_n(z) - c_n g(z) \le 0$ ($n = 1, 2$). Analysis shows that the graph of the function $f_n - c_n g$ touches the abscissa either at two symmetric points or at zero. On satisfying the conditions

$$f_n(z) - c_n g(z) = 0, \quad f_n'(z) - c_n g'(z) = 0$$

we find

$$c_n = \frac{2(1 + (3 - n)\lambda)}{(1 - \sqrt{2}\lambda)^2} \quad \text{when } \frac{1}{\sqrt{2}} < \lambda \le \lambda_n, \quad n = 1, 2$$

$$\begin{aligned} c_1 &= \frac{-2 + \lambda(2 + \sqrt{2})(\lambda + \sqrt{4\lambda^2 - 2\sqrt{2}\lambda^2 - 1})}{\lambda^2 - 1} \quad \text{when } \lambda \ge \lambda_1 \\ c_2 &= \frac{\lambda(2 + \sqrt{2}) + \sqrt{2}(\lambda + \sqrt{4\lambda^2 + 2\sqrt{2}\lambda^2 - 1})}{2(\lambda^2 - 1)} \quad \text{when } \lambda \ge \lambda_2 \end{aligned} \tag{3.7}$$

$$\lambda_1 = -1+\frac{1}{2}\sqrt{2(5+3\sqrt{2})}, \quad \lambda_2 = -1+\sqrt{2}+\frac{1}{2}\sqrt{2(3-\sqrt{2})}$$

It is obvious that $\lambda_n > 1$, and formulae (3.7) therefore define the continuous dependence of c_n as a function of λ when $\lambda > 1/\sqrt{2}$.

Finally, combining formulae (3.5) and (3.6), we obtain the estimates

$$|I_y(x-x_0,y)| \le \frac{c_n}{2}\int_{-\infty}^{\infty} \operatorname{ch} n\mu \exp\{-\Phi_c(\mu)\}d\mu =$$

$$= \frac{c_n}{2}K_n\left(k\cos\left(\frac{\pi}{4}-\varphi\right)r_-\right) \le \frac{c_n}{2}K_n\left(\frac{kr_-}{\sqrt{2}}\right), \quad n = 1,2$$

Applying the transformations indicated above to the integral in the first formula of (3.1), we obtain

$$\partial_x G_0 = \frac{ik}{2\pi}\int_{-\infty}^{\infty} J(\mu)\operatorname{ch}\mu \operatorname{sh}\mu d\mu \tag{3.8}$$

We will again consider the case when $\varphi \in [0, \pi/2)$. On shifting the contour of integration upwards ($\mu \to \mu + i\pi/4$), we find

$$\partial_x G_0 = \frac{k}{2\pi}I_x(x-x_0,y) - \nu e^{\nu y}\begin{cases} \cos l(x-x_0) & \text{when } \lambda > 1 \\ e^{l_*(x-x_0)} & \text{when } \lambda \in (1/\sqrt{2}, 1) \end{cases} \tag{3.9}$$

where

$$I_x(x-x_0,y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{s(\mu)}{g(\mu)}\exp\{-\Phi_c(\mu)\}d\mu$$

$$s(\mu) = \operatorname{ch}2\mu\{(2\lambda-\sqrt{2}\operatorname{ch}\mu)\cos(\Phi_s(\mu))+\sqrt{2}\operatorname{sh}\mu\sin(\Phi_s(\mu))\}$$

Nothing that

$$|s(\mu)| \le \operatorname{ch}2\mu(\sqrt{2}\operatorname{ch}2\mu+2\lambda)$$

we derive the estimates

$$|s(\mu)| \le \operatorname{ch}\mu f_1(\operatorname{ch}\mu), \quad |s(\mu)| \le \operatorname{ch}2\mu f_2(\operatorname{ch}\mu)$$

Similar calculations can be carried out in the case when $\varphi \in (-\pi/2, 0]$ using a shift in the contour of integration $\mu \to \mu - i\pi/4$ in representations (3.2) and (3.8). We therefore finally obtain

$$|I_y(x-x_0,y)| \le \frac{c_n}{2}K_n\left(\frac{kr_-}{\sqrt{2}}\right), \quad |I_x(x-x_0,y)| \le \frac{c_n}{2}K_n\left(\frac{kr_-}{\sqrt{2}}\right) \tag{3.10}$$

4. ESTIMATES OF THE TRAPPED-MODE FREQUENCIES

We will now consider a certain system of bodies $S(Y)$ immersed at a depth

$$Y = \max\{y: (x,y)\in S\}, \quad Y<0$$

The estimates of the derivatives of Green's function and criterion (2.5) enable us to find the subset of the space (ν, k, Y) in which the homogeneous problem (1.1)–(1.3), (1.6) has only a trivial solution.

Using relations (3.3), (3.9) and (3.10), we write

$$\int_S |\partial_{n(x,y)} G_0| ds_{(x,y)} \leq \int_S q(\nu, k, y) ds_{(x,y)}$$

$$q(\nu, k, y) = \begin{cases} q_1(\nu, k, y) \text{ when } 0 < \nu \leq k/\sqrt{2} \\ \min\{q_1(\nu, k, y), q_2(\nu, k, y)\} \text{ when } k/\sqrt{2} < \nu < k \\ q_2(\nu, k, y) \text{ when } \nu > k \end{cases} \tag{4.1}$$

$$q_1(\nu, k, y) = \frac{1}{\pi}\left[\left(\int_0^\infty tH(y, t)dt\right)^2 + \left(\int_0^\infty \sqrt{t^2 + k^2} H(y, t)dt\right)^2\right]^{1/2}$$

$$q_2(\nu, k, y) = \left[\left(\nu^2 + \frac{\nu^4}{|\nu^2 - k^2|}\right)e^{2\nu y} + \frac{k^2}{8\pi^2}\min_{n=1,2}\left\{\left[c_n K_n\left(\frac{k|y|}{\sqrt{2}}\right)\right]^2\right\}\right]^{1/2}$$

The coefficients c_1 and c_2 are defined by formulae (3.7) (where $\lambda = \nu/k$).

The boundary of the subset in the space of the parameters (ν, k, Y) in which the inequalities (2.5) and (4.1) guarantee that there are no non-trivial solutions of the homogeneous problem (1.1)–(1.3), (1.6) is given by the relation

$$\int_{S(Y)} q(\nu, k, y) ds_{(x,y)} = 1 \tag{4.2}$$

The function $q(\nu, k, y)$ decreases monotically as the depth y increases, and the absence of trapped modes is therefore guaranteed for sufficiently large values of Y and, also, for sufficiently large values of the parameter ν. The latter fact follows from the estimate $q_2(\nu, k, y) = O(\nu^{-1})$ when $\nu \to \infty$.

When $\nu/k \to 1$, we have $q_2(\nu, k, y) \sim k^2 e^{ky} |k^2 - \nu^2|^{1/2}$, and the graph of the function $Y(\nu, k)$ which is defined by relation (4.2), has a singularity. The asymptotic form of the dispersion relation

$$\nu a \sim (3\pi)^{-1/2} e^{\nu d} (k^2/\nu^2 - 1)^{1/4}$$

On approaching from below the boundary of the continuous spectrum ($\nu/k \to 1 - 0$) have been obtained in [7] in the case of the natural mode, which is symmetric with respect to x and corresponds to the largest value of the ratio ν/k for a circular cylinder of radius a with its centre at a depth d. From relation (4.2) we obtain

$$\nu a \sim (2\pi)^{-1} e^{\nu d} |k^2/\nu^2 - 1|^{1/2} \quad \text{when} \quad \nu/k \to 1$$

It is obvious that the asymptotic from of the estimate is found to be in accordance with the asymptotic form of the dispersion relation, although not exactly. Here, account must be taken of the fact that the asymptotic form of the dispersion relation is established for the greatest natural frequency located below the boundary of the continuous spectrum and the symmetric natural mode, while Eq. (4.2) serves to estimate all the corresponding modes both in the continuous spectrum as well as outside it.

The results of a numerical investigation of uniqueness using formula (4.2) are shown in Figs 2 and 3, where they are compared with previously obtained results (see [12] and [11] respectively).

The results of calculations for a circular cylinder of radius a with its centre at a depth d are shown in Fig 2. In the case of this geometry, the previously obtained estimates [12] guarantee that there are no trapped modes in the domain defined in the parameter space of the problem by the solutions of the equation

$$K_1(ka) - K_1(k(2d - a)) - \frac{2\nu}{k}\int_0^\infty \frac{e^{(a-2d)\sqrt{t^2+k^2}} dt}{\sqrt{t^2 + k^2} - \nu} = 0 \tag{4.3}$$

For $d/a = 3, 4, 5$, the boundaries of the uniqueness domain for the solution of problem (1.1)–(1.4), which are given by Eq. (4.2), are shown by a solid curve and the boundaries obtained using Eq. (4.3)

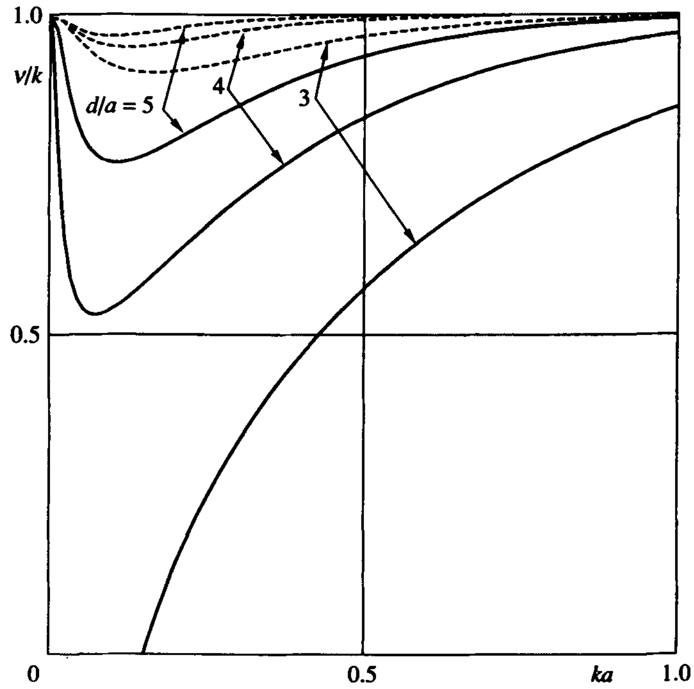

Fig. 2

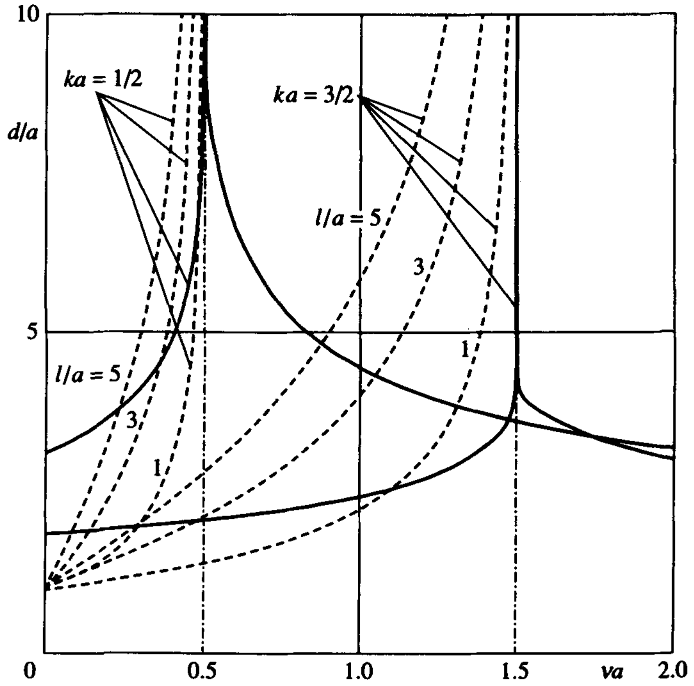

Fig. 3

are shown by a dashed curve. The absence of trapped modes is guaranteed in the domains located under the curves. It should be noted that the previously obtained estimates [12] are more accurate but, at the same time, unlike condition (4.2), they are only suitable for $\nu < k$ and only in the case of a single completely submerged body (since they were obtained using the principle of monotonicity [9]).

The results of a numerical investigation of condition (4.2) and a comparison with the results obtained by Simon [11] for two circular cylinders of radius a with centres at a depth d for a distance between the centres $2l$ and when $ka = 1/2, 3/2$ are shown in Fig. 3. In the case of this geometry when $\nu < k$, Simon's estimates guarantee that there are no trapped modes if

$$\nu k^{-1} \le \sin\{\operatorname{arctg}(dl^{-1}) - \arcsin[a(d^2 + l^2)^{-1/2}]\}$$

The boundaries of the uniqueness domain for the solution of problem (1.1)–(1.4) and (1.1)–(1.5), obtained using formula (4.2), are shown by a solid curve and Simon's results when $l/a = 1, 3, 5$ are shown by a dashed curve. The absence of trapped modes is guaranteed in the domains located under the curves. The estimates (4.2) are independent of the distance between the bodies in a horizontal direction, so that they have an advantage compared with Simon's estimates when there is a large distance between the bodies. At the same time, it should be noted that the results in [11, 12] are only applicable for values of $\nu < k$. There are no analogues of the estimates in this paper for values of $\nu > k$ (in the continuous spectrum of the problem).

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